

TWISTED HOMOLOGY OF QUANTUM $SL(2)$ - PART II

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ABSTRACT. We complete the calculation of the twisted cyclic homology of the quantised coordinate ring $\mathcal{A} = \mathbb{C}_q[SL(2)]$ of $SL(2)$ that we began in [14]. In particular, a nontrivial cyclic 3-cocycle is constructed which also has a nontrivial class in Hochschild cohomology and thus should be viewed as a noncommutative geometry analogue of a volume form.

Dedicated to Prof. K. Schmüdgen on the occasion of his 60th birthday

1. INTRODUCTION

In this paper we finish the computation [14] of the twisted cyclic homology $HC_\bullet^\sigma(\mathcal{A})$ of the quantised coordinate ring $\mathcal{A} = \mathbb{C}_q[SL(2)]$ for $q \in \mathbb{C}$ not a root of unity. This invariant was defined by Kustermans, Murphy and Tuset [24] and modifies Connes' cyclic homology $HC_\bullet(\mathcal{A}) = HC_\bullet^{\text{id}}(\mathcal{A})$ using an automorphism σ of \mathcal{A} . It is computed using the Connes spectral sequence

$$(1) \quad E_{pq}^1 = H_{q-p}(\mathcal{A}, \sigma\mathcal{A}) \Rightarrow HC_{p+q}^\sigma(\mathcal{A}),$$

where $H_\bullet(\mathcal{A}, \cdot)$ is the Hochschild homology of \mathcal{A} and the coefficient bimodule $\sigma\mathcal{A}$ arises from \mathcal{A} by twisting the canonical left action to $a \triangleright b = \sigma(a)b$.

In [14] we were unable to fully compute the spectral sequence in the “no dimension drop” case, that is, for those σ with $H_3(\mathcal{A}, \sigma\mathcal{A}) \neq 0$. The crucial new ingredient we use in this paper are the cup and cap products

$$\begin{aligned} \smile: H^m(\mathcal{A}, \sigma\mathcal{A}) \otimes H^n(\mathcal{A}, \tau\mathcal{A}) &\rightarrow H^{m+n}(\mathcal{A}, \tau\sigma\mathcal{A}), \\ \frown: H_n(\mathcal{A}, \sigma\mathcal{A}) \otimes H_m(\mathcal{A}, \tau\mathcal{A}) &\rightarrow H_{n-m}(\mathcal{A}, \tau^{-1}\sigma\mathcal{A}). \end{aligned}$$

together with the twisted derivations

$$\partial_H^\pm, \partial_E^\pm, \partial_F^\pm \in \bigoplus_{\sigma \in \text{Aut}(\mathcal{A})} H^\bullet(\mathcal{A}, \sigma\mathcal{A})$$

that deform the action of left- and right-invariant vector fields on $SL(2)$. In Section 3 we show that under \smile these derivations generate a q -deformed exterior algebra whose \frown -action allows us to identify nontrivial 2- and 3-cycles. Using this we then compute $HC_\bullet^\sigma(\mathcal{A})$ in Section 4.

Our motivation for [14] was the relation of $HC_\bullet^\sigma(\mathcal{A})$ to Woronowicz's theory of covariant differential calculi [24]. We realised subsequently [15] that the coefficients $\sigma\mathcal{A}$ also arise from Poincaré duality in Hochschild (co)homology: using the general theory of Van den Bergh [32] we showed that

$$(2) \quad H^n(\mathcal{A}, \sigma\mathcal{A}) \simeq H_{3-n}(\mathcal{A}, \sigma_{q^{-2}, 1} \circ \sigma\mathcal{A}) \quad \forall \sigma \in \text{Aut}(\mathcal{A}),$$

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where for $\lambda, \mu \in \mathbb{C} \setminus \{0\}$, $\sigma_{\lambda, \mu} \in \text{Aut}(\mathcal{A})$ is determined by its values

$$\sigma_{\lambda, \mu}(a) = \lambda a, \quad \sigma_{\lambda, \mu}(b) = \mu b, \quad \sigma_{\lambda, \mu}(c) = \mu^{-1} c, \quad \sigma_{\lambda, \mu}(d) = \lambda^{-1} d$$

on the standard generators a, b, c, d of \mathcal{A} . In particular, $\sigma_{q^{-2}, 1}$ is Woronowicz's modular automorphism of the Haar functional on \mathcal{A} , and the “no dimension drop” automorphisms are precisely $\sigma = \sigma_{q^{-N}, 1}$, with $N \in \mathbb{Z}$, $N \geq 2$.

Our main results are summarised in Theorems 1.1 and 1.2 below. Therein,

$$\text{d}\mathcal{A} \in H_3(\mathcal{A}, \sigma_{q^{-2}, 1} \mathcal{A})$$

is the fundamental class in Hochschild homology, that is, it corresponds under (2) to $1 \in H^0(\mathcal{A}, \mathcal{A})$ (identified with the centre of \mathcal{A}), and (2) is given by $\cdot \frown \text{d}\mathcal{A}$ [21]. A cycle in the standard Hochschild complex $C_\bullet(\mathcal{A}, \sigma_{q^{-2}, 1} \mathcal{A}) = \mathcal{A}^{\otimes \bullet + 1}$ representing $\text{d}\mathcal{A}$ is given explicitly in (13).

As a consequence of (2) the Connes spectral sequence (1) stabilises at the second page, and $HC_n^\sigma(\mathcal{A}) \simeq HC_{n+2}^\sigma(\mathcal{A})$ for $n \geq 3$. In this way, we obtain the two periodic cyclic homology groups $HP_{\text{even}}^\sigma(\mathcal{A})$ and $HP_{\text{odd}}^\sigma(\mathcal{A})$ as the limit of the $HC_{2n}^\sigma(\mathcal{A})$ and $HC_{2n+1}^\sigma(\mathcal{A})$, respectively. These groups can be described in the “no dimension drop” case left open in [14] as follows:

THEOREM 1.1. *Let \mathcal{A} be the quantised coordinate ring $\mathbb{C}_q[SL(2)]$, $q \in \mathbb{C}$ not a root of unity, and $\sigma = \sigma_{q^{-N}, 1} \in \text{Aut}(\mathcal{A})$, $N \in \mathbb{Z}$, be determined by*

$$\sigma(a) = q^{-N} a, \quad \sigma(b) = b.$$

Then the σ -twisted periodic cyclic homology of \mathcal{A} is given by

$$\begin{aligned} HP_{\text{even}}^\sigma(\mathcal{A}) &= HC_4^\sigma(\mathcal{A}) = \begin{cases} \mathbb{C}[b^r c^r (\text{d}\mathcal{A} \frown [\partial_H^-])] & : N = 2r + 2, r \geq 0, \\ \mathbb{C}[1] & : \text{otherwise,} \end{cases} \\ HP_{\text{odd}}^\sigma(\mathcal{A}) &= HC_3^\sigma(\mathcal{A}) = \begin{cases} \mathbb{C}[b^r c^r \text{d}\mathcal{A}] & : N = 2r + 2, r \geq 0, \\ \mathbb{C}[b \otimes c] & : \text{otherwise.} \end{cases} \end{aligned}$$

Here classes in $HC_n^\sigma(\mathcal{A})$ are represented by classes in $H_{n-2p}(\mathcal{A}, \sigma \mathcal{A})$, $p \geq 0$ using Connes' spectral sequence $E_{pq}^1 = H_{q-p}(\mathcal{A}, \sigma \mathcal{A}) \Rightarrow HC_{p+q}^\sigma(\mathcal{A})$.

Our second main result, which we prove in Sections 3 and 5, is the explicit construction of a twisted cyclic 3-cocycle ξ that pairs nontrivially with $\text{d}\mathcal{A}$:

THEOREM 1.2. *Define for all $j, k \geq 0$ a functional $\int_{[b^j c^k]} : \mathcal{A} \rightarrow \mathbb{C}$ by*

$$\int_{[b^j c^k]} e_{r,s,t} := \delta_{0,r} \delta_{j,s} \delta_{k,t}, \quad e_{i,j,k} := \begin{cases} a^i b^j c^k & : i \geq 0 \\ d^{-i} b^j c^k & : i < 0 \end{cases}$$

and two linear functionals $C_3(\mathcal{A}, \sigma_{q^{-2}, 1} \mathcal{A}) \rightarrow \mathbb{C}$ by

$$\begin{aligned} \varphi(\cdot) &:= \int_{[1]} \cdot \frown (\partial_H^+ \smile \partial_E^+ \smile \partial_F^+), \\ \eta(\cdot) &:= 2 \int_{[bc]} \cdot \frown (\partial_H^+ \smile (\sigma_{1,1/2} - \text{id}) \smile (\sigma_{1,2} - \text{id})). \end{aligned}$$

Then φ and η are $\sigma_{q^{-2}, 1}$ -twisted Hochschild 3-cocycles, $\xi := \varphi + \eta$ is a $\sigma_{q^{-2}, 1}$ -twisted cyclic 3-cocycle, and $\xi(\text{d}\mathcal{A}) = 1 = \varphi(\text{d}\mathcal{A})$.

The 3-cocycle ξ gives an explicit description in terms of Connes' λ -complex of $HC_{\sigma_{q^{-2},1}}^3(\mathcal{A}) = (HC_3^{\sigma_{q^{-2},1}}(\mathcal{A}))^* \simeq \mathbb{C}$ and should be interpreted from the point of view of noncommutative geometry as an analogue of a volume form, compare e.g. [5], Corollary 35 on p.337 and Connes' orientability axiom [7], Definition 1.233 which requires that the Chern character of a spectral triple should have a nontrivial class in Hochschild cohomology. It had been shown by Masuda, Nakagami and Watanabe that there is no such volume form in the untwisted $HC^3(\mathcal{A})$ [26] but only in $HC^1(\mathcal{A})$, and this "dimension drop" was considered by many authors to be "rather esoteric" [6]. The cocycle ξ solves that mystery, so we expect that there is a spectral triple that realises ξ via a twisted variant of the Connes-Moscovici local index formula, for example as in [9, 28, 30].

As shown by Etingof and Dolgushev [10], a Poincaré-type duality as in (2) holds for formal deformation quantisations of smooth Poisson varieties, and also, as shown by Brown and Zhang, for a large class of Noetherian Hopf algebras that includes in particular the quantised coordinate rings $\mathbb{C}_q[G]$ for all simple algebraic groups G [2]. See also [11, 22] for more information.

Most of the computations in this paper have been verified with the help of the computer algebra system FELIX [1]. The FELIX output is available in electronic form [23], and it can easily be adapted to perform similar computations with other algebras given in terms of generators and relations.

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2. HOCHSCHILD HOMOLOGY

2.1. Background. In Section 2.1, we fix notation and conventions concerning Hochschild homology and the quantised coordinate ring of $SL(2)$. For details and proofs see for example [3, 27, 33] and [19, 20] respectively.

2.1.1. Algebras and bimodules. Throughout this paper "algebra" means unital associative \mathbb{C} -algebra. An unadorned \otimes denotes the tensor product of \mathbb{C} -vector spaces. For an \mathcal{A} -bimodule \mathcal{M} and two automorphisms σ, τ of an algebra \mathcal{A} we denote by ${}_{\sigma}\mathcal{M}_{\tau}$ the bimodule which is \mathcal{M} as vector space with bimodule structure $x \blacktriangleright y \blacktriangleleft z := \sigma(x) \triangleright y \triangleleft \tau(z)$, $x, z \in \mathcal{A}, y \in \mathcal{M}$, where $\triangleleft, \triangleright$ are the original actions on \mathcal{M} . Note the bimodule isomorphisms

$$\sigma'({}_{\sigma}\mathcal{M}_{\tau})_{\tau'} \simeq {}_{\sigma\sigma'}\mathcal{M}_{\tau\tau'}, \quad \mathcal{M} \otimes_{\mathcal{A}} {}_{\sigma}\mathcal{N} \simeq {}_{\sigma\sigma^{-1}}\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}, \quad \mathcal{A}_{\sigma^{-1}} \simeq {}_{\sigma}\mathcal{A}.$$

2.1.2. Hochschild homology. The Hochschild homology groups of an algebra \mathcal{A} with coefficients in an \mathcal{A} -bimodule \mathcal{M} are

$$H_n(\mathcal{A}, \mathcal{M}) = \mathrm{Tor}_n^{\mathcal{A}^e}(\mathcal{M}, \mathcal{A}),$$

where $\mathcal{A}^e := \mathcal{A} \otimes \mathcal{A}^{\text{op}}$ is the enveloping algebra of \mathcal{A} (so \mathcal{A}^e -modules are just \mathcal{A} -bimodules). They can be computed using the canonical bar resolution of \mathcal{A} as an \mathcal{A}^e -module, and are then realised as the simplicial homology of the simplicial \mathbb{C} -vector space $C_\bullet(\mathcal{A}, \mathcal{M}) := \mathcal{M} \otimes \mathcal{A}^{\otimes \bullet}$ whose structure maps are

$$\begin{aligned} b_0 &: a_0 \otimes \dots \otimes a_n \mapsto a_0 \triangleleft a_1 \otimes a_2 \otimes \dots \otimes a_n, \\ b_i &: a_0 \otimes \dots \otimes a_n \mapsto a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n, \quad 0 < i < n, \\ b_n &: a_0 \otimes \dots \otimes a_n \mapsto a_n \triangleright a_0 \otimes \dots \otimes a_{n-1}, \\ s_i &: a_0 \otimes \dots \otimes a_n \mapsto a_0 \otimes \dots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \dots \otimes a_n, \quad 0 \leq i \leq n. \end{aligned}$$

That is $H_\bullet(\mathcal{A}, \mathcal{M})$ is (isomorphic to) the homology of the chain complex $C_\bullet(\mathcal{A}, \mathcal{M})$ whose boundary map is given by

$$b := \sum_{i=0}^n (-1)^i b_i.$$

In the sequel we will often write $b(a_0, \dots, a_n)$ instead of $b(a_0 \otimes \dots \otimes a_n)$.

If \mathcal{A} is the coordinate ring $\mathbb{C}[X]$ of a smooth affine variety, then $H_\bullet(\mathcal{A}, \mathcal{A})$ can be identified canonically with the Kähler differentials (algebraic differential forms) on X (Hochschild-Kostant-Rosenberg theorem).

2.1.3. The normalised complex. For any simplicial \mathbb{C} -vector space C ,

$$D := \text{span}\{\text{im } s_i\} \subset C$$

is a contractible subcomplex with respect to b , so the canonical map

$$C \rightarrow \bar{C} := C/D$$

is a quasi-isomorphism of complexes, and working with the so-called normalised complex (\bar{C}_\bullet, b) simplifies many computations.

For $C = C(\mathcal{A}, \mathcal{M})$ from the previous section, we have $\bar{C}_n = \mathcal{M} \otimes \bar{\mathcal{A}}^{\otimes n}$, where $\bar{\mathcal{A}} := \mathcal{A}/\mathbb{C}$. So informally speaking in the computation of Hochschild homology we can neglect all elementary tensors with a tensor component being equal to a multiple of $1 \in \mathcal{A}$.

2.1.4. Quantum $SL(2)$. For the remainder of Section 2, $q \in \mathbb{C}$ denotes a fixed nonzero parameter, which we assume is not a root of unity. Furthermore, \mathcal{A} is throughout the quantised coordinate algebra $\mathbb{C}_q[SL(2)]$ of $SL(2)$, that is the algebra generated by symbols a, b, c, d with relations

$$\begin{aligned} ab &= qba, & ac &= qca, & bc &= cb, & bd &= qdb, & cd &= qdc, \\ ad - qbc &= 1, & da - q^{-1}bc &= 1. \end{aligned}$$

It follows from the defining relations that the elements

$$(3) \quad e_{i,j,k} := \begin{cases} a^i b^j c^k & : i \geq 0 \\ d^{-i} b^j c^k & : i < 0 \end{cases} \quad i \in \mathbb{Z}, j, k \in \mathbb{N}$$

form a vector space basis of \mathcal{A} .

For $\lambda, \mu \in \mathbb{C}^\times$ there are unique automorphisms $\sigma_{\lambda,\mu}, \tau_{\lambda,\mu}$ of \mathcal{A} with

$$\begin{aligned} \sigma_{\lambda,\mu}(a) &= \lambda a, & \sigma_{\lambda,\mu}(b) &= \mu b, & \sigma_{\lambda,\mu}(c) &= \mu^{-1} c, & \sigma_{\lambda,\mu}(d) &= \lambda^{-1} d, \\ \tau_{\lambda,\mu}(a) &= \lambda a, & \tau_{\lambda,\mu}(b) &= \mu^{-1} c, & \tau_{\lambda,\mu}(c) &= \mu b, & \tau_{\lambda,\mu}(d) &= \lambda^{-1} d, \end{aligned}$$

and all automorphisms of \mathcal{A} are of this form (see [19]). For later use, we compute for all $\sigma_{\lambda,\mu}$ the twisted commutators

$$\begin{aligned}
e_{i,j,k}a - \lambda a e_{i,j,k} &= (q^{-j-k} - \lambda)e_{i+1,j,k} \\
&\quad + \begin{cases} 0 & : i \geq 0 \\ (q^{-j-k-1} - \lambda q^{-1-2i})e_{i+1,j+1,k+1} & : i < 0, \end{cases} \\
(4) \quad e_{i,j,k}b - \mu b e_{i,j,k} &= (1 - \mu q^{-i})e_{i,j+1,k}, \\
e_{i,j,k}c - \mu^{-1}c e_{i,j,k} &= (1 - \mu^{-1}q^{-i})e_{i,j,k+1}, \\
e_{i,j,k}d - \lambda^{-1}d e_{i,j,k} &= (q^{j+k} - \lambda^{-1})e_{i-1,j,k} \\
&\quad + \begin{cases} 0 & : i \leq 0 \\ (q^{j+k+1} - \lambda^{-1}q^{1-2i})e_{i-1,j+1,k+1} & : i > 0 \end{cases}.
\end{aligned}$$

Finally, recall (see [20]) that the standard Hopf algebra structure on \mathcal{A} admits a so-called universal r-form $\mathbf{r} : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathbb{C}$. This can be used to define a braiding

$$\Psi : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}, \quad x \otimes y \mapsto \mathbf{r}(y_{(1)}, x_{(1)})y_{(2)} \otimes x_{(2)}\mathbf{r}(S(y_{(3)}), x_{(3)}),$$

where $x \mapsto x_{(1)} \otimes x_{(2)}$ is the coproduct in Sweedler notation and $S : \mathcal{A} \rightarrow \mathcal{A}$ is the antipode. This braiding should be considered as a quantum analogue of the tensor flip and is used in the standard way to define

$$\begin{aligned}
x \wedge y &:= (\text{id} - \Psi)(x \otimes y), \\
(5) \quad x \wedge y \wedge z &:= (\text{id} - \Psi_{1,2} - \Psi_{2,3} + \Psi_{2,3} \circ \Psi_{1,2} + \Psi_{1,2} \circ \Psi_{2,3} \\
&\quad - \Psi_{1,2} \circ \Psi_{2,3} \circ \Psi_{1,2})(x \otimes y \otimes z)
\end{aligned}$$

where $\Psi_{2,3} := \text{id} \otimes \Psi$ and $\Psi_{1,2} := \Psi \otimes \text{id}$. On generators, we have

$$\begin{bmatrix} \mathbf{r}(a,a) & \mathbf{r}(a,b) & \mathbf{r}(a,c) & \mathbf{r}(a,d) \\ \mathbf{r}(b,a) & \mathbf{r}(b,b) & \mathbf{r}(b,c) & \mathbf{r}(b,d) \\ \mathbf{r}(c,a) & \mathbf{r}(c,b) & \mathbf{r}(c,c) & \mathbf{r}(c,d) \\ \mathbf{r}(d,a) & \mathbf{r}(d,b) & \mathbf{r}(d,c) & \mathbf{r}(d,d) \end{bmatrix} = q^{-1/2} \begin{bmatrix} q & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & q - q^{-1} & 0 & 0 \\ 1 & 0 & 0 & q \end{bmatrix},$$

where $q^{-1/2}$ is a fixed solution of $z^2 = q^{-1}$, and

$$\begin{aligned}
\Psi(a \otimes a) &= a \otimes a, & \Psi(a \otimes b) &= q^{-1}b \otimes a + (1 - q^{-2})a \otimes b, \\
\Psi(a \otimes c) &= qc \otimes a, & \Psi(a \otimes d) &= d \otimes a + (q - q^{-1})c \otimes b, \\
\Psi(b \otimes a) &= q^{-1}a \otimes b, & \Psi(b \otimes b) &= b \otimes b, & \Psi(b \otimes c) &= c \otimes b, \\
\Psi(b \otimes d) &= qd \otimes b, & \Psi(c \otimes a) &= qa \otimes c + (1 - q^2)c \otimes a, \\
\Psi(c \otimes b) &= b \otimes c - (q - q^{-1})^2c \otimes b + (q - q^{-1})a \otimes d + (q^{-1} - q)d \otimes a, \\
\Psi(c \otimes c) &= c \otimes c, & \Psi(c \otimes d) &= q^{-1}d \otimes c + (1 - q^{-2})c \otimes d, \\
\Psi(d \otimes a) &= a \otimes d - (q - q^{-1})c \otimes b, & \Psi(d \otimes b) &= qb \otimes d + (1 - q^2)d \otimes b, \\
\Psi(d \otimes c) &= q^{-1}c \otimes d, & \Psi(d \otimes d) &= d \otimes d.
\end{aligned}$$

2.2. Results. Here we recall from [14] the description of $H_n(\mathcal{A}, \sigma\mathcal{A})$ for $\mathcal{A} = \mathbb{C}_q[SL(2)]$ and $\sigma = \sigma_{\lambda,\mu}$, but we simplify the presentation and also correct some errors. Throughout, elements of $H_n(\mathcal{A}, \sigma\mathcal{A})$ are represented in the normalised Hochschild complex $\bar{C}_\bullet(\mathcal{A}, \sigma\mathcal{A})$.

2.2.1. $n = 0$. $H_0(\mathcal{A}, \sigma\mathcal{A})$ is easily computed directly, using the canonical complex $C_\bullet(\mathcal{A}, \sigma\mathcal{A})$. Since

$$(6) \quad x \otimes yz = xy \otimes z + \sigma(z)x \otimes y - \mathbf{b}(x, y, z),$$

the boundary operator \mathbf{b} on $C_1(\mathcal{A}, \sigma\mathcal{A})$ satisfies

$$\mathbf{b}(x, yz) = \mathbf{b}(xy, z) + \mathbf{b}(\sigma(z)x, y),$$

so its image is spanned by $\mathbf{b}(e_{i,j,k}, a)$, $\mathbf{b}(e_{i,j,k}, b)$, $\mathbf{b}(e_{i,j,k}, c)$ and $\mathbf{b}(e_{i,j,k}, d)$, that is, by the twisted commutators (4). This yields a description of $H_0(\mathcal{A}, \sigma\mathcal{A})$ that can be summarised in compact form as follows: we first define

$$(7) \quad \omega_{r,i} := b^i c^{r-i}, \quad S(\lambda) := \begin{cases} \mathbb{N} \setminus \{N - 2r : r \geq 1\} & : \lambda = q^{-N}, N \geq 2 \\ \mathbb{N} & : \text{otherwise.} \end{cases}$$

Then the following set of homology classes is a vector space basis of $H_0(\mathcal{A}, \sigma\mathcal{A})$:

$$(8) \quad \begin{aligned} & \{[a^i], [d^i] : i \geq 0 \mid \lambda = 1\} \\ & \cup \{[b^j], [c^j] : j \in S(\lambda) \mid \mu = 1\} \\ & \cup \{[\omega_{N,i}] : 1 \leq i \leq N - 1 \mid \lambda = q^{-N}, N \geq 2, \mu = 1\} \\ & \cup \{[e_{M,N,0}], [e_{-M,0,N}] \mid \lambda = q^{-N}, N > 0, \mu = q^M, M \neq 0\}. \end{aligned}$$

We use the convention that $[x^0] := [1]$, for any $x \in \{a, b, c, d\}$, and multiple occurrences of any $[y]$ are counted only once. To ensure the notation is clear, consider the case $\mu = 1$, $\lambda = q^{-N}$, $N \geq 2$ (Case 2 in [14]). Then (8) gives a basis

$$(9) \quad \{[b^j], [c^j] : j \in S(\lambda)\} \cup \{[\omega_{N,i}] : 1 \leq i \leq N - 1\},$$

with $S(\lambda) = \mathbb{N} \setminus \{N - 2, N - 4, N - 6, \dots\}$, exactly as in [14], p343.

Although $H_0(\mathcal{A}, \sigma\mathcal{A})$ was computed correctly in Section 4.3 of [14], the overview on p.328-329 therein claimed that $H_0(\mathcal{A}, \sigma\mathcal{A})$ is infinite-dimensional only for $\mu = 1$, whereas it should correctly read for $\mu = 1$ or $\lambda = 1$.

2.2.2. $n = 1$. We also used $C_\bullet(\mathcal{A}, \sigma\mathcal{A})$ to compute $H_1(\mathcal{A}, \sigma\mathcal{A})$. By (6), $H_1(\mathcal{A}, \sigma\mathcal{A})$ is generated by the classes of linear combinations of $e_{i,j,k} \otimes a$, $e_{i,j,k} \otimes b$, $e_{i,j,k} \otimes c$ and $e_{i,j,k} \otimes d$. These are mapped by \mathbf{b} to the twisted commutators (4), and it is simple to compute which linear combinations of these tensors defines a cycle and which of these are homologous to each other (see [14]). This gives the following vector space basis of $H_1(\mathcal{A}, \sigma\mathcal{A})$:

$$(10) \quad \begin{aligned} & \{[(1 - \mu^{-1})d \otimes a + (q - q^{-1})b \otimes c] \mid \lambda = 1\} \\ & \cup \{[a^i \otimes a], [d^i \otimes d] : i \geq 0 \mid \lambda = 1\} \\ & \cup \{[b^{j-1} \otimes b], [c^{j-1} \otimes c] : j \in S(\lambda) \mid \mu = 1\} \\ & \cup \{[\omega_{N-1,i} \otimes b], [\omega_{N-1,i+1} \otimes c] : 0 \leq i \leq N - 2 \mid \lambda = q^{-N}, N \geq 2, \mu = 1\} \\ & \cup \{[a^{M-1}b^N \otimes a], [a^M b^{N-1} \otimes b] \mid \lambda = q^{-N}, \mu = q^M, M, N > 0\} \\ & \cup \{[d^M c^{N-1} \otimes c], [d^{M-1} c^N \otimes d] \mid \lambda = q^{-N}, \mu = q^M, M, N > 0\} \\ & \cup \{[d^{M-1}b^N \otimes d], [d^M b^{N-1} \otimes b] \mid \lambda = q^{-N}, \mu = q^{-M}, M, N > 0\} \\ & \cup \{[a^M c^{N-1} \otimes c], [a^{M-1} c^N \otimes a] \mid \lambda = q^{-N}, \mu = q^{-M}, M, N > 0\}. \end{aligned}$$

Here $S(\lambda)$ and $\omega_{r,i}$ are as in (7) and we use the formal notation (neither b nor c are invertible elements)

$$[c^{-1} \otimes c] := [b \otimes c], \quad [b^{-1} \otimes b] := [c \otimes b]$$

which appears in the above set for $\mu = 1$ except when $\lambda = q^{-N}$ with $N = 2r$, $r > 0$, but should be counted only once:

LEMMA 2.1. *If $\lambda \neq q^{-2}$, then $[b \otimes c] = -\mu^{-1}[c \otimes b]$ for all $\mu \in \mathbb{C}$.*

Proof. This follows from $[1 \otimes bc] = [b \otimes c] + \mu^{-1}[c \otimes b]$ (a special case of (6)) together with $b(1 \otimes (a \otimes d - \lambda^{-1}d \otimes a + (1 - \lambda^{-1}) \otimes 1)) = (\lambda^{-1}q^{-1} - q) \otimes bc$. \square

2.2.3. $n = 2$. In higher degrees, working with the canonical complex is no longer feasible, but we showed in [14], Proposition 4.1 that the trivial left \mathcal{A} -module \mathbb{C} (on which \mathcal{A} acts by the counit ε of the standard Hopf algebra structure) admits a noncommutative Koszul resolution of the form

$$(11) \quad 0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}^3 \longrightarrow \mathcal{A}^3 \longrightarrow \mathcal{A} \longrightarrow 0$$

with morphisms given by the matrices

$$\begin{pmatrix} c & -b & q^{-2}a - 1 \end{pmatrix}, \quad \begin{pmatrix} b & 1 - q^{-1}a & 0 \\ c & 0 & 1 - q^{-1}a \\ 0 & c & -b \end{pmatrix}, \quad \begin{pmatrix} a - 1 \\ b \\ c \end{pmatrix}$$

that operate by right multiplication on row vectors. Then we computed $H_n(\mathcal{A}, {}_{\sigma}\mathcal{A})$ for $n \geq 2$, using the alternative derived functor description of $H_{\bullet}(\mathcal{A}, \mathcal{M})$ for Hopf algebras as in Feng and Tsygan [12].

The result is that $H_2(\mathcal{A}, {}_{\sigma}\mathcal{A}) = 0$ except when $\lambda = q^{-N}$, $N > 0$, and in this case one has

$$\dim_{\mathbb{C}} H_2(\mathcal{A}, {}_{\sigma}\mathcal{A}) = \begin{cases} 2(N-1) & : \mu = 1 \\ 2 & : \mu = q^{\pm M}, M > 0 \\ 0 & : \mu \notin q^{\mathbb{Z}} \end{cases}$$

A linear basis is given by

$$(12) \cup \begin{aligned} & \{[\omega_2(N-2, i)], [\omega'_2(N-2, i)] : 0 \leq i \leq N-2 \mid \mu = 1\} \\ & \cup \{[a^{M-1}b^{N-1} \otimes (b \wedge a)], [d^{M-1}c^{N-1} \otimes (d \wedge c)] \mid \mu = q^M, M > 0\}, \\ & \cup \{[a^{M-1}c^{N-1} \otimes (a \wedge c)], [d^{M-1}b^{N-1} \otimes (b \wedge d)] \mid \mu = q^{-M}, M > 0\}, \end{aligned}$$

where $x \wedge y$ was defined in (5) and

$$\begin{aligned} \omega_2(r, i) &:= \omega_{r,i}(bc \otimes (a \wedge d) - bd \otimes (a \wedge c) + \\ & \quad da \otimes (b \wedge c) - q^{-1}ca \otimes (b \wedge d)), \\ \omega'_2(r, i) &:= \omega_{r,i} \otimes (b \wedge c). \end{aligned}$$

These formulae differ from [14] by a correction we found after S. Launois pointed out to us an inconsistency between our results in [14] and [15] (see [25]): the second half of the first sentence after Proposition 4.10 on p.349 in [14] is incorrect, the generators given for $\mu = 1$ are all linearly independent in homology.

2.2.4. $n = 3$. We have $H_3(\mathcal{A}, \sigma\mathcal{A}) = 0$ except when $\lambda = q^{-N}$, $N \geq 2$, $\mu = 1$, and in this case

$$\dim_{\mathbb{C}} H_3(\mathcal{A}, \sigma_{q^{-N}, 1}\mathcal{A}) = N - 1,$$

with a basis given by the classes of

$$\begin{aligned} \omega_3(N-2, i) &:= \omega_{N-2, i}(-qd \otimes (b \wedge a \wedge c) + c \otimes (b \wedge a \wedge d)), \\ &= b^i c^{N-2-i} (d \otimes (-qb \otimes a \otimes c + a \otimes b \otimes c + q^2 b \otimes c \otimes a \\ &\quad - a \otimes c \otimes b - q^2 c \otimes b \otimes a + qc \otimes a \otimes b) \\ &\quad + c \otimes (b \otimes a \otimes d - q^{-1} a \otimes b \otimes d - b \otimes d \otimes a \\ &\quad - (q - q^{-1}) b \otimes c \otimes b + a \otimes d \otimes b + qd \otimes b \otimes a - d \otimes a \otimes b)), \end{aligned}$$

for $0 \leq i \leq N-2$. We abbreviate

$$(13) \quad d\mathcal{A} := [\omega_3(0, 0)].$$

We remark that in the explicit formula for $b \wedge a \wedge d$ given in [14] the last term $-(q - q^{-1})b^i c^{N-1-i} \otimes b \otimes c \otimes b$ was missing, and that the above defines a cycle in the unnormalised complex $C_{\bullet}(\mathcal{A}, \sigma_{q^{-N}, 1}\mathcal{A})$, not just in $\bar{C}_{\bullet}(\mathcal{A}, \sigma_{q^{-N}, 1}\mathcal{A})$.

2.2.5. $n > 3$. Since the resolution (11) has length 3, $H_n(\mathcal{A}, \sigma\mathcal{A}) = 0$ for $n > 3$.

3. HOCHSCHILD COHOMOLOGY

3.1. Background. Section 3.1 recalls products and Poincaré duality in Hochschild (co)homology (see e.g. [3, 21, 29, 32]), as well as twisted derivations of $\mathbb{C}_q[SL(2)]$ that arise from the left and right actions of its Hopf dual.

3.1.1. Hochschild cohomology. Let \mathcal{A} be a unital associative \mathbb{C} -algebra. The Hochschild cohomology $H^n(\mathcal{A}, \mathcal{M}) := \text{Ext}_{\mathcal{A}^e}^n(\mathcal{A}, \mathcal{M})$ of \mathcal{A} with coefficients in an \mathcal{A} -bimodule \mathcal{M} can be computed as the cohomology of the cochain complex $C^{\bullet}(\mathcal{A}, \mathcal{M})$ of \mathbb{C} -linear maps $\varphi : \mathcal{A}^{\otimes \bullet} \rightarrow \mathcal{M}$ with coboundary map given by

$$\begin{aligned} (b\varphi)(a_1, \dots, a_{n+1}) &:= a_1 \triangleright \varphi(a_2, \dots, a_{n+1}) \\ &\quad + \sum_{j=1}^n (-1)^j \varphi(a_1, \dots, a_j a_{j+1}, \dots, a_{n+1}) \\ &\quad + (-1)^{n+1} \varphi(a_1, \dots, a_n) \triangleleft a_{n+1}. \end{aligned}$$

This presentation of cocycles yields the standard identification

$$H^0(\mathcal{A}, \mathcal{M}) \simeq \{m \in \mathcal{M} \mid a \triangleright m = m \triangleleft a \text{ for all } a\},$$

and of $H^1(\mathcal{A}, \mathcal{M})$ with the space of derivations

$$(14) \quad \partial : \mathcal{A} \rightarrow \mathcal{M}, \quad \partial(xy) = x \triangleright \partial(y) + \partial(x) \triangleleft y$$

modulo inner derivations (those of the form $x \mapsto x \triangleright m - m \triangleleft x$, $m \in \mathcal{M}$).

3.1.2. *The cup product.* The cup product

$$\smile : H^m(\mathcal{A}, \mathcal{M}) \otimes H^n(\mathcal{A}, \mathcal{N}) \rightarrow H^{m+n}(\mathcal{A}, \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N})$$

is defined on the level of cochains by

$$(\varphi \smile \psi)(a_1, \dots, a_{m+n}) := \varphi(a_1, \dots, a_m) \otimes_{\mathcal{A}} \psi(a_{m+1}, \dots, a_{m+n}),$$

where $\varphi \in C^m(\mathcal{A}, \mathcal{M})$, $\psi \in C^n(\mathcal{A}, \mathcal{N})$. Since

$$\mathbf{b}(\varphi \smile \psi) = (\mathbf{b}\varphi) \smile \psi + (-1)^m \varphi \smile (\mathbf{b}\psi)$$

the cup product is well-defined on the level of cohomology. As a special case, we obtain for $\sigma, \tau \in \text{Aut}(\mathcal{A})$ a map

$$(15) \quad H^m(\mathcal{A}, {}_{\sigma}\mathcal{A}) \otimes H^n(\mathcal{A}, {}_{\tau}\mathcal{A}) \rightarrow H^{m+n}(\mathcal{A}, {}_{\tau \circ \sigma}\mathcal{A})$$

given on cochains by

$$(16) \quad (\varphi \smile \psi)(a_1, \dots, a_{m+n}) = \tau(\varphi(a_1, \dots, a_m)) \psi(a_{m+1}, \dots, a_{m+n}).$$

Thus for any monoid $G \subset \text{Aut}(\mathcal{A})$ we obtain an $\mathbb{N} \times G$ -graded algebra

$$\Lambda_G^{\bullet}(\mathcal{A}) := \bigoplus_{n \in \mathbb{N}, \sigma \in G} H^n(\mathcal{A}, {}_{\sigma}\mathcal{A}).$$

We call the subalgebra $\Lambda_G^0(\mathcal{A})$ the G -twisted centre of \mathcal{A} and the elements of the $\Lambda_G^0(\mathcal{A})$ -bimodule $\Lambda_G^1(\mathcal{A})$ (or rather the cocycles representing them) the G -twisted derivations of \mathcal{A} . Obviously, G could be replaced by any monoid of bimodules, but we will not need this in the present paper.

Note that in degree 0, (16) reduces to the *opposite* product of \mathcal{A} ,

$$(17) \quad x \smile y = \tau(x)y = yx, \quad x \in H^0(\mathcal{A}, {}_{\sigma}\mathcal{A}), \quad y \in H^0(\mathcal{A}, {}_{\tau}\mathcal{A}),$$

and that for $z \in H^0(\mathcal{A}, {}_{\sigma}\mathcal{A})$ and $\partial \in C^1(\mathcal{A}, {}_{\tau}\mathcal{A})$ we have

$$(18) \quad \begin{aligned} (z \smile \partial)(x) &= \sigma(\partial(x))z = z\partial(x), \\ (\partial \smile z)(x) &= \tau(z)\partial(x), \\ \Rightarrow \partial \smile z &= \tau(z) \smile \partial. \end{aligned}$$

Finally, for twisted derivations $\partial \in C^1(\mathcal{A}, {}_{\sigma}\mathcal{A})$, $\partial' \in C^1(\mathcal{A}, {}_{\tau}\mathcal{A})$ we have

$$\begin{aligned} & (\partial \smile \partial' + (\sigma^{-1} \circ \partial' \circ \sigma) \smile \partial)(x, y) \\ &= \tau(\partial(x))\partial'(y) + \partial'(\sigma(x))\partial(y) \\ &= \partial'(\partial(x)y) - \partial'(\partial(x))y + \partial'(\sigma(x)\partial(y)) - \tau(\sigma(x))\partial'(\partial(y)) \\ &= \partial'(\partial(xy)) - \partial'(\partial(x))y - \tau(\sigma(x))\partial'(\partial(y)) \\ &= -\mathbf{b}(\partial' \circ \partial)(x, y), \end{aligned}$$

so in cohomology

$$(19) \quad [\partial] \smile [\partial'] = -[\sigma^{-1} \circ \partial' \circ \sigma] \smile [\partial] \in H^2(\mathcal{A}, {}_{\tau \circ \sigma}\mathcal{A}).$$

3.1.3. *The cap product.* The duality between Hochschild homology and cohomology results from the cap product pairing

$$\frown : H_n(\mathcal{A}, \mathcal{M}) \otimes H^m(\mathcal{A}, \mathcal{N}) \rightarrow H_{n-m}(\mathcal{A}, \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}), \quad m \leq n$$

defined on (co)chains by evaluation,

$$(a_0 \otimes \dots \otimes a_n) \frown \varphi = a_0 \otimes_{\mathcal{A}} \varphi(a_1, \dots, a_m) \otimes a_{m+1} \otimes \dots \otimes a_n.$$

For $m = n$, the pairing \frown becomes the duality pairing from [27], Section 1.5.9 after identifying

$$H_0(\mathcal{A}, \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}) = \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N} \otimes_{\mathcal{A}^e} \mathcal{A} \simeq \mathcal{M} \otimes_{\mathcal{A}^e} \mathcal{N}.$$

Taking $\mathcal{N} = \mathcal{M}^* = \text{Hom}_{\mathbb{C}}(\mathcal{M}, \mathbb{C})$ and composing with the canonical evaluation map $\mathcal{M} \otimes_{\mathcal{A}^e} \mathcal{M}^* \rightarrow \mathbb{C}$ gives the duality pairing

$$H_n(\mathcal{A}, \mathcal{M}) \otimes H^n(\mathcal{A}, \mathcal{M}^*) \rightarrow \mathbb{C}.$$

By the universal coefficient theorem, this yields an isomorphism $H^n(\mathcal{A}, \mathcal{M}^*) \simeq (H_n(\mathcal{A}, \mathcal{M}))^*$. In this way a Hochschild cocycle $\varphi \in C^n(\mathcal{A}, \mathcal{M}^*)$ will usually be viewed as a \mathbb{C} -linear map $\mathcal{M} \otimes \mathcal{A}^{\otimes n} \rightarrow \mathbb{C}$.

For any $G \subset \text{Aut}(\mathcal{A})$, the cap product endows

$$\Omega_{\bullet}^G(\mathcal{A}) := \bigoplus_{n \in \mathbb{N}, \sigma \in G} H_n(\mathcal{A}, \sigma^{-1} \mathcal{A})$$

with the structure of an $\mathbb{N} \times G$ -graded (right) module over $\Lambda_G^{\bullet}(\mathcal{A})$ (here we use the convention that a homogeneous element of a graded ring lowers the degree of an element of a homogeneous module by its degree). Remember to take into account the identification $\sigma \mathcal{A} \otimes_{\mathcal{A}} \tau \mathcal{A} \rightarrow \tau \circ \sigma \mathcal{A}$: explicitly, the action of $\varphi \in C^m(\mathcal{A}, \tau \mathcal{A})$ on $a_0 \otimes \dots \otimes a_n \in C_n(\mathcal{A}, \sigma \mathcal{A})$ is given by

$$(a_0 \otimes \dots \otimes a_n) \frown \varphi = \tau(a_0) \varphi(a_1, \dots, a_m) \otimes a_{m+1} \otimes \dots \otimes a_n \in C_{n-m}(\mathcal{A}, \tau \circ \sigma \mathcal{A}).$$

In particular, the cap product with a twisted central element $z \in H^0(\mathcal{A}, \sigma \mathcal{A})$ is simply given by multiplication from the left,

$$(20) \quad (a_0 \otimes \dots \otimes a_n) \frown z = \sigma(a_0) z \otimes \dots \otimes a_n = z a_0 \otimes \dots \otimes a_n.$$

The cap product is also the source of Poincaré-type dualities in Hochschild (co)homology. In particular, we have for $\mathcal{A} = \mathbb{C}_q[SL(2)]$:

THEOREM 3.1. [15, 21] *For any bimodule \mathcal{N} over $\mathcal{A} := \mathbb{C}_q[SL(2)]$, the map*

$$(21) \quad \cdot \frown \text{d}\mathcal{A} : H^n(\mathcal{A}, \mathcal{N}) \rightarrow H_{3-n}(\mathcal{A}, \sigma_{q^{-2}, 1} \mathcal{N})$$

is an isomorphism of \mathbb{C} -vector spaces.

3.1.4. *Twisted primitive elements in the Hopf dual of $\mathbb{C}_q[SL(2)]$.* For the rest of Section 3, we focus on $\mathcal{A} = \mathbb{C}_q[SL(2)]$ and recall (see e.g. [18, 20] and references therein) that the Hopf dual of the standard Hopf algebra structure on \mathcal{A} contains a Hopf subalgebra \mathcal{U} with generators H, K, K^{-1}, E, F and relations

$$\begin{aligned} KK^{-1} &= K^{-1}K = 1, & KEK^{-1} &= q^2E, & KFK^{-1} &= q^{-2}F, \\ [E, F] &= \frac{K - K^{-1}}{q - q^{-1}}, & [H, K] &= 0, & [H, E] &= 2E, & [H, F] &= -2F. \end{aligned}$$

This is the standard Drinfeld-Jimbo quantised universal enveloping algebra $U_q(\mathfrak{sl}(2))$ extended by the unquantised functional H (so when working with formal deformations we would have $K = e^{\hbar H}$, $q = e^{\hbar}$).

The dual pairing of \mathcal{U} and \mathcal{A} gives two commuting left and right actions of \mathcal{U} on \mathcal{A} , and the operators assigned by these actions to H, EK^{-1} and F are (twisted) derivations which we denote by

$$\begin{aligned}\partial_H^+ : a, b, c, d &\mapsto -a, b, -c, d, & \partial_H^- : a, b, c, d &\mapsto -a, -b, c, d \in C^1(\mathcal{A}, \mathcal{A}), \\ \partial_E^+ : a, b, c, d &\mapsto qb, 0, qd, 0, & \partial_F^+ : a, b, c, d &\mapsto 0, a, 0, c \in C^1(\mathcal{A}, \sigma_{q, q^{-1}}\mathcal{A}), \\ \partial_E^- : a, b, c, d &\mapsto 0, 0, q^{-1}a, q^{-1}b, & \partial_F^- : a, b, c, d &\mapsto c, d, 0, 0 \in C^1(\mathcal{A}, \sigma_{q, q}\mathcal{A}).\end{aligned}$$

Due to the Leibniz rule (14), this determines the derivations uniquely.

3.2. Results. Here we compute the twisted centre and twisted derivations of $\mathbb{C}_q[SL(2)]$. Then we show how their cap product action on the twisted Hochschild homology groups can be used to determine the homology class of a given 2- or 3-cycle.

3.2.1. Twisted central elements. The twisted centre $\Lambda^0 := \Lambda_{\text{Aut}(\mathcal{A})}^0(\mathcal{A})$ of \mathcal{A} is a (commutative) polynomial ring in two indeterminates:

LEMMA 3.2. *There is an isomorphism of graded algebras*

$$\Lambda^0 = \bigoplus_{N \geq 0} H^0(\mathcal{A}, \sigma_{q^{-N}, 1}\mathcal{A}) \simeq \mathbb{C}[b, c].$$

Proof. By Poincaré duality (Theorem 3.1) and the computation of $H_3(\mathcal{A}, \sigma\mathcal{A})$ recalled in Section 2.2.4, $H^0(\mathcal{A}, \sigma\mathcal{A})$ vanishes except when $\sigma = \sigma_{q^{-N}, 1}$, $N \geq 0$, and in this case it has dimension $N + 1$ over \mathbb{C} . The monomials $\omega_{N, i} = b^i c^{N-i}$, $0 \leq i \leq N$, are $N + 1$ linearly independent elements of $H^0(\mathcal{A}, \sigma_{q^{-N}, 1}\mathcal{A})$ and hence form a vector space basis. Finally, (17) gives $\omega_{r, i} \smile \omega_{s, j} = \omega_{r, i}\omega_{s, j} = \omega_{r+s, i+j}$ since b and c commute. \square

The cap product action of Λ^0 gives additional structure to the results of our computations of $H_\bullet(\mathcal{A}, \sigma\mathcal{A})$. For example, the direct sum of the nontrivial $H_3(\mathcal{A}, \sigma\mathcal{A})$ forms a free module of rank one over Λ^0 with generator $d\mathcal{A}$. Similarly we will see below that applying twisted derivations to $d\mathcal{A}$ leads for example to the 2-dimensional $H_2(\mathcal{A}, \sigma_{q^{-N}, q^{\pm M}}\mathcal{A})$ in (12).

3.2.2. Detecting nontrivial 2-cycles. For large n and m it is typically difficult to decide whether a given cycle $c \in C_n(\mathcal{A}, \sigma\mathcal{A})$ and cocycle $\varphi \in C^m(\mathcal{A}, \tau\mathcal{A})$ have nontrivial classes in (co)homology. A sufficient criterion is that $c \smile \varphi \in C_{n-m}(\mathcal{A}, \tau \circ \sigma\mathcal{A})$ is nontrivial in homology, and this may be easy to verify for small $n - m$. We now give an example of this kind for $\mathcal{A} = \mathbb{C}_q[SL(2)]$, whose result is used later in the computation of twisted cyclic homology and also to determine $\Lambda_{\text{Aut}(\mathcal{A})}^1(\mathcal{A})$.

LEMMA 3.3. *Abbreviate $\partial := \frac{1}{2}(\partial_H^+ + \partial_H^-)$ and $\partial' := -\partial_H^-$. Then:*

$$\begin{aligned}[\omega_2(N-2, i)] \smile [\partial] &= [\omega_2'(N-2, i)] \smile [\partial'] = [\omega_{N-1, i+1} \otimes c] + [\omega_{N-1, i} \otimes b], \\ [\omega_2'(N-2, i)] \smile [\partial] &= [\omega_2(N-2, i)] \smile [\partial'] = 0.\end{aligned}$$

Proof. For $\sigma = \sigma_{q^{-2},1}$ we have in $\bar{C}_1(\mathcal{A}, \sigma\mathcal{A})$:

$$\begin{aligned} \mathbf{b}(bc \otimes a \otimes d) &= q^{-2}abc \otimes d - qbc \otimes bc + q^2dbc \otimes a, \\ \mathbf{b}(bc \otimes b \otimes c) &= b^2c \otimes c - bc \otimes bc + bc^2 \otimes b, \\ \mathbf{b}(ca \otimes (d \wedge b)) &= (q^3 - q)bc^2 \otimes b, \\ \mathbf{b}(ba \otimes (d \wedge c)) &= (q - q^{-1})b^2c \otimes c. \end{aligned}$$

Using this, we compute directly that

$$\begin{aligned} [\omega_2(0,0)] \frown [\partial_H^+] &= 2[-q^{-2}abc \otimes d - q^2dbc \otimes a + qbc^2 \otimes b \\ &\quad + q^{-1}b^2c \otimes c + c \otimes b + b \otimes c] \\ &= 2(q^{-1} - q)[\omega_{3,2} \otimes c] + 2[\omega_{1,1} \otimes c] + 2[\omega_{1,0} \otimes b] \\ &= 2[\omega_{1,1} \otimes c] + 2[\omega_{1,0} \otimes b], \end{aligned}$$

that $[\omega_2(0,0)] \frown [\partial_H^-] = 0$, and that

$$[\omega_2'(0,0)] \frown [\partial_H^+] = -[\omega_2'(0,0)] \frown [\partial_H^-] = [c \otimes b + b \otimes c].$$

The claim follows by $\Lambda_{\text{Aut}(\mathcal{A})}^0(\mathcal{A})$ -linearity of the products. \square

COROLLARY 3.4. *The $2(N+1)$ cohomology classes*

$$[\omega_{N,i}] \smile [\partial], \quad [\omega_{N,i}] \smile [\partial'], \quad 0 \leq i \leq N,$$

are linearly independent in $H^1(\mathcal{A}, \sigma_{q^{-N},1}\mathcal{A})$.

3.2.3. Twisted derivations. We now describe the twisted derivations (modulo inner derivations) $\Lambda^1 := \Lambda_{\text{Aut}(\mathcal{A})}^1(\mathcal{A})$ as a bimodule over the twisted centre Λ^0 , and study their relations under the cup product.

First, note that for all $i > 0$ the cochains

$$\begin{aligned} C^1(\mathcal{A}, \sigma_{q,q^{-i}}\mathcal{A}) &\ni \partial_E^+ \smile d^{i-1} : a, b, c, d \mapsto qd^{i-1}b, 0, qd^i, 0, \\ C^1(\mathcal{A}, \sigma_{q,q^{-i}}\mathcal{A}) &\ni \partial_F^+ \smile a^{i-1} : a, b, c, d \mapsto 0, a^i, 0, a^{i-1}c, \\ C^1(\mathcal{A}, \sigma_{q,q^i}\mathcal{A}) &\ni \partial_E^- \smile a^{i-1} : a, b, c, d \mapsto 0, 0, q^{-1}a^i, q^{-1}a^{i-1}b, \\ C^1(\mathcal{A}, \sigma_{q,q^i}\mathcal{A}) &\ni \partial_F^- \smile d^{i-1} : a, b, c, d \mapsto d^{i-1}c, d^i, 0, 0, \end{aligned}$$

are (twisted) derivations, although $a^{i-1}, d^{i-1} \notin \Lambda^0$. The point is that for example $\partial_E^+(\mathcal{A}) \subset \mathcal{B}$, where $\mathcal{B} \subset \mathcal{A}$ is the subalgebra generated by b, d , and we have $d^{i-1} \in \Lambda_{\text{Aut}(\mathcal{B})}^0(\mathcal{B})$ with twisting automorphism extending to the whole of \mathcal{A} . This implies the claim, which of course can also be verified directly. We denote the classes of these derivations in cohomology by $\partial_i^+, \partial_{-i}^+, \partial_i^-, \partial_{-i}^-$, respectively, and define $\partial_0^\pm := [\partial_H^\pm]$.

LEMMA 3.5. *As a left Λ^0 -module, Λ^1 is generated by $\{\partial_i^\pm, i \in \mathbb{Z}\}$, and*

$$\begin{aligned} (22) \quad [b] \smile \partial_{-i}^\pm &= 0, \quad [c] \smile \partial_i^\pm = 0, \quad i > 1, \\ \partial_i^\pm \smile [b^j c^k] &= q^{\pm i(j-k)} [b^j c^k] \smile \partial_i^\pm, \quad i \in \mathbb{Z}, j, k \in \mathbb{N}, \\ \partial_i^\varepsilon \smile \partial_j^\delta &= -q^{\delta j + \varepsilon |i| \text{sgn}(j)} \partial_j^\delta \smile \partial_i^\varepsilon, \quad i, j \in \mathbb{Z}, \varepsilon, \delta \in \{-1, +1\}. \end{aligned}$$

Proof. The second and third relations in (22) are computed directly from (18) and (19). Next, Section 2.2.3 and (21) give

$$\dim_{\mathbb{C}} H^1(\mathcal{A}, \sigma_{\lambda, \mu} \mathcal{A}) = \begin{cases} 2(N+1) & : \lambda = q^{-N}, N > 0, \mu = 1, \\ 2 & : \lambda = q^{-N}, N > 0, \mu = q^M, M \neq 0, \\ 0 & : \text{otherwise.} \end{cases}$$

Corollary 3.4 implies that $\{[\omega_{N,i}] \smile \partial_0^{\pm} \mid 0 \leq i \leq N\}$ is a linearly independent subset of $H^1(\mathcal{A}, \sigma_{q^{-N}, 1} \mathcal{A})$, so it is a basis for dimension reasons. The ∂_i^{\pm} , $i \neq 0$, are treated similarly. For example, we can use that for $i, j > 0$

$$[\omega'_2(0,0)] \smile (\partial_i^{\pm} \smile \partial_{-j}^{\pm}) = -q^{\pm(1-i)}[e_{\pm(j-i),0,0}] \in H_0(\mathcal{A}, \sigma_{1, q^{\mp(i+j)}} \mathcal{A}),$$

and that by the third line in (18),

$$\partial_i^{\varepsilon} \smile \partial_j^{\delta} = q^{(\text{sgn}(i)+\text{sgn}(j))(\varepsilon|i|+\delta|j|)} \partial_i^{\varepsilon} \smile \partial_j^{\delta},$$

so $\partial_i^{\varepsilon} \smile \partial_j^{\delta} = 0$ except when $\text{sgn}(j) = -\text{sgn}(i)$ or $\delta = -\varepsilon, |j| = |i|$. \square

3.2.4. Detecting nontrivial 3-cycles. We present computations similar to Section 3.2.2, this time acting on $\omega_3(0,0)$ to decide whether 3-cycles are nontrivial.

LEMMA 3.6. *In $H_1(\mathcal{A}, \sigma_{q^{-N}, 1} \mathcal{A})$, we have*

$$[\omega_3(N-2, i)] \smile ([\partial_H^+] \smile [\partial_H^-]) = 2([\omega_{N-1, i+1} \otimes c] + [\omega_{N-1, i} \otimes b]).$$

In particular, $\cdot \smile ([\partial_H^+] \smile [\partial_H^-]) : \Omega_3^{\text{Aut}(\mathcal{A})}(\mathcal{A}) \rightarrow \Omega_1^{\text{Aut}(\mathcal{A})}(\mathcal{A})$ is injective.

Proof. Directly, $[\omega_3(0,0)] \smile [\partial_H^-] = -[\omega_2(0,0)]$ and $\omega_3(r, i) = \omega_3(0,0) \smile \omega_{r,i}$. The result then follows from Lemmas 3.3 and 3.5. \square

As we shall see below, this Lemma is strong enough to detect nontrivial homology classes. However, it would be easier (and more standard) to apply a further twisted derivation to obtain 0-cycles whose classes in homology are even simpler to control than those of 1-cycles. While this works for the fundamental class $\mathbf{d}\mathcal{A} = [\omega_3(0,0)]$, it is unfortunately not possible for all the other generators $[\omega_3(r, i)]$, $r > 0$: the orbit of $\mathbf{d}\mathcal{A}$ under the cap product

action of Λ^1 is determined completely by Lemma 3.5 and the relations

$$\begin{aligned}
\omega_3(0,0) \frown (\partial_H^+ \smile \partial_E^+ \smile \partial_F^+) &= (q^{-1} - q)bc + 1, \\
\omega_3(0,0) \frown (\partial_H^+ \smile \partial_E^+ \smile \partial_H^-) &= 2(q^4 - 1)db^2c - 2qdb, \\
\omega_3(0,0) \frown (\partial_H^+ \smile \partial_E^+ \smile \partial_E^-) &= (q - q^{-3})b^3c - q^{-2}b^2, \\
\omega_3(0,0) \frown (\partial_H^+ \smile \partial_E^+ \smile \partial_F^-) &= (q - q^5)d^2bc + d^2, \\
\omega_3(0,0) \frown (\partial_H^+ \smile \partial_F^+ \smile \partial_H^-) &= 2(q - q^{-3})abc^2 - 2ac, \\
\omega_3(0,0) \frown (\partial_H^+ \smile \partial_F^+ \smile \partial_E^-) &= (q^{-1} - q^{-5})a^2bc - a^2, \\
\omega_3(0,0) \frown (\partial_H^+ \smile \partial_F^+ \smile \partial_F^-) &= (q^{-1} - q^3)bc^3 + (2 - q^2)c^2, \\
\omega_3(0,0) \frown (\partial_H^+ \smile \partial_H^- \smile \partial_E^-) &= 2(q^{-4} - q^{-2})ab^2c + 2q^{-1}ab, \\
\omega_3(0,0) \frown (\partial_H^+ \smile \partial_H^- \smile \partial_F^-) &= 2(q - q^3)dbc^2 + 2dc, \\
\omega_3(0,0) \frown (\partial_H^+ \smile \partial_E^- \smile \partial_F^-) &= 2(1 - q^2)b^2c^2 + 2q^{-1}bc + 1, \\
\omega_3(0,0) \frown (\partial_E^+ \smile \partial_F^+ \smile \partial_H^-) &= 2(q^{-4} - q^2)b^2c^2 + (2q^{-3} - q + q^{-1})bc + 1, \\
\omega_3(0,0) \frown (\partial_E^+ \smile \partial_F^+ \smile \partial_E^-) &= (q^{-7} - q^{-1})ab^2c + q^{-4}ab, \\
\omega_3(0,0) \frown (\partial_E^+ \smile \partial_F^+ \smile \partial_F^-) &= (q^4 - q^{-2})dbc^2 - q^{-1}dc, \\
\omega_3(0,0) \frown (\partial_E^+ \smile \partial_H^- \smile \partial_E^-) &= (q - q^{-3})b^3c - q^{-2}b^2, \\
\omega_3(0,0) \frown (\partial_E^+ \smile \partial_H^- \smile \partial_F^-) &= (q^5 - q)d^2bc - d^2, \\
\omega_3(0,0) \frown (\partial_E^+ \smile \partial_E^- \smile \partial_F^-) &= (q^5 - q)db^2c - db, \\
\omega_3(0,0) \frown (\partial_F^+ \smile \partial_H^- \smile \partial_E^-) &= (q^{-5} - q^{-1})a^2bc + a^2, \\
\omega_3(0,0) \frown (\partial_F^+ \smile \partial_H^- \smile \partial_F^-) &= (q^{-1} - q^3)bc^3 + (2 - q^2)c^2, \\
\omega_3(0,0) \frown (\partial_F^+ \smile \partial_E^- \smile \partial_F^-) &= (q^{-2} - q^2)abc^2 + q^{-1}ac, \\
\omega_3(0,0) \frown (\partial_H^- \smile \partial_E^- \smile \partial_F^-) &= 1,
\end{aligned}$$

which hold on the level of chains in the normalised Hochschild complex. By inspection we now obtain:

LEMMA 3.7. *We have*

$$\begin{aligned}
d\mathcal{A} \frown [\partial_H^+ \smile \partial_E^+ \smile \partial_F^+] &= [1] \in H_0(\mathcal{A}, \sigma_{1,q^{-2}}\mathcal{A}), \\
[\omega_3(r,r)] \frown [\partial_H^+ \smile \partial_E^- \smile \partial_E^+] &= [b^{r+2}] \in H_0(\mathcal{A}, \sigma_{q^{-r},1}\mathcal{A}), \\
[\omega_3(r,0)] \frown [\partial_H^- \smile \partial_F^- \smile \partial_F^+] &= [c^{r+2}] \in H_0(\mathcal{A}, \sigma_{q^{-r},1}\mathcal{A}),
\end{aligned}$$

but for all $\partial_1, \partial_2, \partial_3 \in \Lambda^1$, $z \in \Lambda^0$, and $0 < i < r$ we have

$$[\omega_3(r,i)] \frown [z \smile \partial_1 \smile \partial_2 \smile \partial_3] = 0.$$

Proof. The first three statements are obtained by direct computation using the above description of the orbit of $\omega_3(0,0)$ under the cap product action of the twisted derivations $\{\partial_H^\pm, \partial_E^\pm, \partial_F^\pm\}$, the fact that $\omega_3(r,i) = \omega_3(0,0) \frown \omega_{r,i}$ and the commutation relations (22). For the final statement, by Lemma 3.5 we can assume without loss of generality that $\partial_1, \partial_2, \partial_3 \in \{\partial_H^\pm, \partial_E^\pm, \partial_F^\pm\}$. The same Lemma implies that we then have

$$\begin{aligned}
&[\omega_3(r,i)] \frown [z \smile \partial_1 \smile \partial_2 \smile \partial_3] \\
&= [\omega_3(0,0)] \frown [\omega_{r,i} \smile z \smile \partial_1 \smile \partial_2 \smile \partial_3] \\
&= [\omega_3(0,0)] \frown [\partial_1 \smile \partial_2 \smile \partial_3 \smile z']
\end{aligned}$$

for some $z' \in \Lambda^0$. The homology class of $\omega_3(0,0) \frown (\partial_1 \smile \partial_2 \smile \partial_3)$ can be read off for these ∂_i from the list above (using also the commutation relations (22) of the ∂_i), and whenever the result is nonzero, then Lemma 3.5 gives $[\omega_{r,i} \smile \partial_1 \smile \partial_2 \smile \partial_3] = 0$ for $0 < i < r$, which implies the claim. \square

We can now begin the proof of Theorem 1.2. We wish to compose the action of $[\partial_H^+ \smile \partial_E^+ \smile \partial_F^+]$ above with a suitable twisted trace

$$\int : \mathcal{A} \rightarrow \mathbb{C}, \quad \int xy = \int \sigma_{1,q^{-2}}(y)x.$$

to obtain a numerical invariant of $\mathbf{d}\mathcal{A}$. For $\mathcal{A} = \mathbb{C}_q[SL(2)]$, the complete list of twisted traces can be given as follows: for any element $[e_{i,j,k}]$ of our basis (8) of $H_0(\mathcal{A}, \sigma\mathcal{A})$ define a linear functional $\int_{[e_{i,j,k}]}$ by

$$\int_{[e_{i,j,k}]} e_{r,s,t} := \delta_{i,r} \cdot \begin{cases} \sum_{n=0}^{\infty} \delta_{s,j+n} \delta_{t,k+n} (-q)^n \frac{1-q^{j+k}\lambda}{1-q^{j+k+2n}\lambda} & : i = 0, jk = 0 \\ \delta_{s,j} \delta_{t,k} & : \text{otherwise.} \end{cases}$$

These then descend to linearly independent functionals on $H_0(\mathcal{A}, \sigma\mathcal{A})$ that are dual to the basis (8), $\int_{[e_{i,j,k}]} [e_{r,s,t}] = \delta_{i,r} \delta_{j,s} \delta_{k,t}$ for $[e_{i,j,k}], [e_{r,s,t}] \in (8)$.

If $\varphi \in C^n(\mathcal{A}, \sigma\mathcal{A})$ is an n -cocycle and $\int \in C^0(\mathcal{A}, (\tau\mathcal{A})^*)$ is a twisted trace, using $\sigma\mathcal{A} \otimes (\tau\mathcal{A})^* \simeq \sigma(\tau\mathcal{A})^* \simeq (\tau\mathcal{A}\sigma)^* \simeq (\sigma^{-1}\circ\tau\mathcal{A})^*$, $\varphi \smile \int$ can be identified with a functional on $H_n(\mathcal{A}, \sigma^{-1}\circ\tau\mathcal{A})$. In particular Lemma 3.7 gives:

COROLLARY 3.8. *Let $\int_{[1]} : \mathcal{A} \rightarrow \mathbb{C}$ be the $\sigma_{1,q^{-2}}$ -twisted trace given by*

$$\int_{[1]} e_{r,s,t} := \delta_{0,r} \delta_{0,s} \delta_{0,t}.$$

Then

$$\int_{[1]} \mathbf{d}\mathcal{A} \frown [\partial_H^+ \smile \partial_E^+ \smile \partial_F^+] = 1.$$

Hence the linear functional

$$(23) \quad \varphi(a_0, a_1, a_2, a_3) := \int_{[1]} \sigma_{q^2, q^{-2}}(a_0 \partial_H^+(a_1)) \sigma_{q, q^{-1}}(\partial_E^+(a_2)) \partial_F^+(a_3)$$

is a Hochschild 3-cocycle with a nontrivial class in $H^3(\mathcal{A}, (\sigma_{q^{-2}, 1}\mathcal{A})^*)$ that is dual to the fundamental class $\mathbf{d}\mathcal{A}$ in the sense that $\varphi(\mathbf{d}\mathcal{A}) = 1$.

We will complete the proof of Theorem 1.2 in Section 5.

4. CYCLIC HOMOLOGY

4.1. Background. In Section 4.1 we recall the definition of the twisted cyclic homology of an algebra. For more background see for example [27, 33].

4.1.1. Paracyclic objects and their homology. Paracyclic objects [13] (say in an abelian category) slightly generalise Connes' cyclic objects [4]:

DEFINITION 4.1. *A paracyclic object is a simplicial object $(C_\bullet, \mathbf{b}_\bullet, \mathbf{s}_\bullet)$ equipped with morphisms $\mathbf{t} : C_n \rightarrow C_n$ that satisfy (on C_n)*

$$\mathbf{b}_i \mathbf{t} = -\mathbf{t} \mathbf{b}_{i-1}, \quad \mathbf{s}_i \mathbf{t} = -\mathbf{t} \mathbf{s}_{i-1}, \quad \mathbf{b}_0 \mathbf{t} = (-1)^n \mathbf{b}_n, \quad \mathbf{s}_0 \mathbf{t} = (-1)^n \mathbf{s}_n, \quad 1 \leq i \leq n.$$

The difference with cyclic objects is that $T := t^{n+1}$ is not required to be the identity id . However, it can be directly verified that T commutes with all the paracyclic generators t, b_i, s_j . As a consequence, a cyclic object can be attached to any paracyclic object by passing to the coinvariants $C/\text{im}(\text{id} - T)$ of T . In well-behaved cases, there is no loss of homological information in this step - for example, we have ([14], Proposition 2.1):

LEMMA 4.2. *If C is a paracyclic \mathbb{C} -vector space and T is diagonalisable, then $(C, b) \rightarrow (C/\text{im}(\text{id} - T), b)$ is a quasi-isomorphism.*

Just as for cyclic objects, for any paracyclic object define

$$N := \sum_{i=0}^n t^i, \quad s := (-1)^{n+1} t s_n, \quad B := (\text{id} - t) s N,$$

all acting on C_n , and as in the cyclic case we have

$$(24) \quad b(\text{id} - t) = (\text{id} - t)b', \quad b'N = Nb, \quad sb' + b's = \text{id},$$

where $b' := \sum_{i=0}^{n-1} (-1)^i b_i$. The operator B satisfies in general

$$BB = (\text{id} - T)(\text{id} - t)ssN, \quad bB + Bb = \text{id} - T.$$

The cyclic homology $HC_\bullet(C)$ of a paracyclic object is the total homology of the bicomplex $(E_{pq}^0 := C_{q-p}/\text{im}(\text{id} - T), b, B)$, $p, q \geq 0$ (so it only depends on the cyclic object associated to C). If p, q take arbitrary values in \mathbb{Z} and we consider the direct product total complex we obtain the periodic cyclic homology $HP_\bullet(C)$. As usual we will use the spectral sequence arising from filtering E^0 by columns as the main tool for computing $HC_\bullet(C)$.

Recall finally that $B(D) \subset D$, where $D := \text{span}\{\text{im } s_i\}$ is the degenerate part of C . Therefore, B descends to the normalised complex \bar{C}_\bullet , where it takes the simpler form $B = sN$ since $ts = (-1)^{n+1} t s_n = -s_0 t$. Therefore, we work with \bar{C}_\bullet throughout our explicit computations below.

4.1.2. *Twisted cyclic homology of an algebra \mathcal{A} .* If \mathcal{A} is an algebra and $\sigma \in \text{Aut}(\mathcal{A})$, then the simplicial object $C_\bullet(\mathcal{A}, \sigma\mathcal{A})$ is in fact paracyclic [24] with

$$t : a_0 \otimes \dots \otimes a_n \mapsto (-1)^n \sigma(a_n) \otimes a_0 \otimes a_1 \otimes \dots \otimes a_{n-1}.$$

For this paracyclic object, B is given on $\bar{C}_\bullet(\mathcal{A}, \sigma\mathcal{A})$ by

$$B : a_0 \otimes \dots \otimes a_n \mapsto 1 \otimes \sum_{i=0}^n (-1)^{ni} \sigma(a_{n-i+1}) \otimes \dots \otimes \sigma(a_n) \otimes a_0 \otimes \dots \otimes a_{n-i}.$$

Following [24], we denote by $C_\bullet^\sigma(\mathcal{A}) := C_\bullet(\mathcal{A}, \sigma\mathcal{A})/\text{im}(\text{id} - T)$ the associated cyclic object and by $HH_\bullet^\sigma(\mathcal{A})$ and $HC_\bullet^\sigma(\mathcal{A})$ its simplicial and cyclic homology, respectively. By Lemma 4.2 we have $H_\bullet(\mathcal{A}, \sigma\mathcal{A}) \simeq HH_\bullet^\sigma(\mathcal{A})$ if σ is diagonalisable. This is crucial in the calculation of $HC_\bullet^\sigma(\mathcal{A})$ since $H_\bullet(\mathcal{A}, \sigma\mathcal{A})$ is computable via its derived functor description, while $HH_\bullet^\sigma(\mathcal{A})$ is the first page of the Connes spectral sequence $E \Rightarrow HC^\sigma(\mathcal{A})$. For an example of an algebra and a non-diagonalisable σ for which the map is not an isomorphism, see [16], Example 3.10.

In the case $\sigma = \text{id}$, $HC_\bullet^\sigma(\mathcal{A})$ reduces to standard cyclic homology $HC_\bullet(\mathcal{A})$ [4, 31]. If $\mathcal{A} = \mathbb{C}[X]$ for a smooth affine variety, then the Hochschild-Kostant-Rosenberg isomorphism identifies B with Cartan's exterior differential, and

the Connes spectral sequence stabilises at E^2 , giving

$$(25) \quad HP_n(\mathbb{C}[X]) \simeq \bigoplus_{i \geq 0} H_{\text{deRham}}^{2i+n}(X),$$

where the right hand side is the even and odd algebraic de Rham cohomology of X with coefficients in \mathbb{C} , see e.g. [5, 27].

We mention finally that $HC_{\bullet}^{\sigma}(\mathcal{A})$ can also be viewed as a special case of Hopf-cyclic homology, see [8, 17].

4.2. Results. We now prove Theorem 1.1 by computing the twisted cyclic homology $HC_{\bullet}^{\sigma}(\mathcal{A})$ of $\mathcal{A} = \mathbb{C}_q[SL(2)]$ for $\sigma = \sigma_{q^{-N}, 1}$, $N \geq 2$. To do so we show that the spectral sequence $E \Rightarrow HC^{\sigma}(\mathcal{A})$ stabilises at the second page, so the result can be read off from E^2 . For the computation for all other $\sigma_{\lambda, \mu}$, see [14].

Lemma 4.2 gives $HH_{\bullet}^{\sigma}(\mathcal{A}) \simeq H_{\bullet}(\mathcal{A}, {}_{\sigma}\mathcal{A})$, which is reflected by the fact that all the generators of $H_{\bullet}(\mathcal{A}, {}_{\sigma}\mathcal{A})$ listed in Section 2.2 are invariant under $\mathbf{T} = \sigma \otimes \dots \otimes \sigma$. From now on we suppress the distinction between $HH_{\bullet}^{\sigma}(\mathcal{A})$ and $H_{\bullet}(\mathcal{A}, {}_{\sigma}\mathcal{A})$. We will compute $\mathbf{B} : HH_n^{\sigma}(\mathcal{A}) \rightarrow HH_{n+1}^{\sigma}(\mathcal{A})$ on the vector space bases from Section 2.2. It will then be possible to read off directly the (co)homology E^2 , and it is then immediate that E^{\bullet} stabilises at E^2 . As before, we work throughout in the normalised complex $\bar{C}_{\bullet}(\mathcal{A}, {}_{\sigma}\mathcal{A})$.

4.2.1. E_{pp}^2 and $HH_1^{\sigma}(\mathcal{A})/\text{im } \mathbf{B}$. For $n = 0$, the action of \mathbf{B} on the basis (9) of $HH_0^{\sigma}(\mathcal{A})$ is given by

$$\begin{aligned} [1] &\mapsto 0, & [x^i] &\mapsto i[x^{i-1} \otimes x], & x = b, c, & i \geq 1, \\ [\omega_{N,i}] &\mapsto i[\omega_{N-1,i-1} \otimes b] + (N-i)[\omega_{N-1,i} \otimes c], & 0 \leq i \leq N, \end{aligned}$$

(see [14], Lemma 2.2). Comparing this with the basis (10) of $HH_1^{\sigma}(\mathcal{A})$ gives:

LEMMA 4.3. *For $p > 0$ we have:*

$$\begin{aligned} E_{pp}^2 &= \begin{cases} \mathbb{C}[1] & : 0 \in S(\lambda), \\ 0 & : 0 \notin S(\lambda), \end{cases} \\ HH_1^{\sigma}(\mathcal{A})/\text{im } \mathbf{B} &= \bigoplus_{i=0}^{N-2} \mathbb{C}[\omega_{N-1,i} \otimes b] \oplus \begin{cases} \mathbb{C}[b \otimes c] & : 0 \in S(\lambda), \\ 0 & : 0 \notin S(\lambda), \end{cases} \end{aligned}$$

where we identify elements of $HH_1^{\sigma}(\mathcal{A})$ with their classes in $HH_1^{\sigma}(\mathcal{A})/\text{im } \mathbf{B}$.

Recall that $S(\lambda)$ was defined in (7), and $0 \notin S(\lambda)$ if and only if $N \geq 2$ and N is even.

4.2.2. E_{pp+1}^2 and $HH_2^{\sigma}(\mathcal{A})/\text{im } \mathbf{B}$. The basis (10) is mapped by \mathbf{B} to

$$\begin{aligned} [b^{j-1} \otimes b], [c^{j-1} \otimes c] &\mapsto 0, & j > 0, & j \in S(\lambda), \\ [b \otimes c] &\mapsto [1 \otimes (b \wedge c)], & 0 \in S(\lambda), \\ [\omega_{N-1,i} \otimes b] &\mapsto -(N-1-i)[\omega_2'(N-2, i)], & 0 \leq i \leq N-2, \\ [\omega_{N-1,i} \otimes c] &\mapsto i[\omega_2'(N-2, i-1)], & 1 \leq i \leq N-1. \end{aligned}$$

Now note that Lemmas 2.1 and 3.3 imply $[1 \otimes (b \wedge c)] = 0$ for $0 \in S(\lambda)$, because in this case

$$[1 \otimes (b \wedge c)] \frown [\partial] = 0, \quad [1 \otimes (b \wedge c)] \frown [\partial'] = -[b \otimes c] - [c \otimes b] = 0$$

Comparing with our descriptions of $HH_1^\sigma(\mathcal{A})/\text{im } \mathbf{B}$ from Lemma 4.3 and of $HH_2^\sigma(\mathcal{A})$ given in Section 2.2.3, this yields:

LEMMA 4.4. *For $p \geq 1$ we have:*

$$\begin{aligned} E_{pp+1}^2 &= \begin{cases} \mathbb{C}[b \otimes c] & : 0 \in S(\lambda), \\ 0 & : 0 \notin S(\lambda), \end{cases} \\ HH_2^\sigma(\mathcal{A})/\text{im } \mathbf{B} &= \bigoplus_{i=0}^{N-2} \mathbb{C}[\omega_2(N-2, i)]. \end{aligned}$$

4.2.3. E_{pp+2}^2 and $E_{pp+3}^2 = HH_3^\sigma(\mathcal{A})/\text{im } \mathbf{B}$. This involves a lengthier computation, so we state the result first.

LEMMA 4.5. *For $\mathbf{B} : HH_2^\sigma(\mathcal{A}) \rightarrow HH_3^\sigma(\mathcal{A})$ we have*

$$(26) \quad \mathbf{B}([\omega_2(r, i)]) = (2i - r)[\omega_3(r, i)], \quad 0 \leq i \leq r.$$

Therefore, we have for $p \geq 2$

$$\begin{aligned} E_{pp+2}^2 &= \begin{cases} \mathbb{C}[\omega_2(2r, r)] & : N = 2r + 2, \ r \geq 0, \\ 0 & : \text{otherwise,} \end{cases} \\ E_{pp+3}^2 &= HH_3^\sigma(\mathcal{A})/\text{im } \mathbf{B} = \begin{cases} \mathbb{C}[\omega_3(2r, r)] & : N = 2r + 2, \ r \geq 0, \\ 0 & : \text{otherwise.} \end{cases} \end{aligned}$$

Proof. We prove (26), the second part is then immediate. We have

$$\begin{aligned} &\mathbf{B}(\omega_2(r, i)) \\ &= \mathbf{B}(\omega_{r,i}(bc \otimes (a \otimes d - d \otimes a - (q - q^{-1})c \otimes b) - bd \otimes (a \otimes c - qc \otimes a) \\ &\quad + da \otimes (b \otimes c - c \otimes b) - q^{-1}ca \otimes (b \otimes d - qd \otimes b))) \\ &= 1 \otimes (\omega_{r+2,i+1} \otimes a \otimes d + q^{r+2}d \otimes \omega_{r+2,i+1} \otimes a + a \otimes d \otimes \omega_{r+2,i+1} \\ &\quad - \omega_{r+2,i+1} \otimes d \otimes a - q^{-r-2}a \otimes \omega_{r+2,i+1} \otimes d - d \otimes a \otimes \omega_{r+2,i+1} \\ &\quad - (q - q^{-1})(\omega_{r+2,i+1} \otimes c \otimes b + b \otimes \omega_{r+2,i+1} \otimes c + c \otimes b \otimes \omega_{r+2,i+1}) \\ &\quad - \omega_{r,i}bd \otimes a \otimes c - c \otimes \omega_{r,i}bd \otimes a - q^{-r-2}a \otimes c \otimes \omega_{r,i}bd \\ &\quad + q\omega_{r,i}bd \otimes c \otimes a + q^{-r-1}a \otimes \omega_{r,i}bd \otimes c + q^{-r-1}c \otimes a \otimes \omega_{r,i}bd \\ &\quad + \omega_{r,i}da \otimes b \otimes c + c \otimes \omega_{r,i}da \otimes b + b \otimes c \otimes \omega_{r,i}da - \omega_{r,i}da \otimes c \otimes b \\ &\quad - b \otimes \omega_{r,i}da \otimes c - c \otimes b \otimes \omega_{r,i}da - q^{-1}\omega_{r,i}ca \otimes b \otimes d \\ &\quad - q^{r+1}d \otimes \omega_{r,i}ca \otimes b - q^{r+1}b \otimes d \otimes \omega_{r,i}ca + \omega_{r,i}ca \otimes d \otimes b \\ &\quad + b \otimes \omega_{r,i}ca \otimes d + q^{r+2}d \otimes b \otimes \omega_{r,i}ca) \end{aligned}$$

Apply $\frown \partial$, where $\partial = \frac{1}{2}(\partial_H^+ + \partial_H^-)$. This gives

$$\begin{aligned} &(\mathbf{B}(\omega_2(r, i))) \frown \partial \\ &= q^{r+2}d \otimes \omega_{r+2,i+1} \otimes a - a \otimes d \otimes \omega_{r+2,i+1} + q^{-r-2}a \otimes \omega_{r+2,i+1} \otimes d \\ &\quad - d \otimes a \otimes \omega_{r+2,i+1} - \omega_{r,i}bd \otimes a \otimes c + q^{-r-2}a \otimes c \otimes \omega_{r,i}bd \\ &\quad + q\omega_{r,i}bd \otimes c \otimes a - q^{-r-1}a \otimes \omega_{r,i}bd \otimes c + q^{-1}\omega_{r,i}ca \otimes b \otimes d \\ &\quad - q^{r+1}d \otimes \omega_{r,i}ca \otimes b - \omega_{r,i}ca \otimes d \otimes b + q^{r+2}d \otimes b \otimes \omega_{r,i}ca. \end{aligned}$$

Now apply $\smile \frac{1}{2}(\partial_H^+ - \partial_H^-)$. This gives

$$\begin{aligned}
& ((B(\omega_2(r, i))) \smile \partial) \smile \frac{1}{2}(\partial_H^+ - \partial_H^-) \\
&= (2i - r)q^{r+2}d\omega_{r+2, i+1} \otimes a + (2i - r)q^{-r-2}a\omega_{r+2, i+1} \otimes d \\
&\quad - q^{-r-2}ac \otimes \omega_{r+1, i+1}d - q\omega_{r+1, i+1}dc \otimes a \\
&\quad - (2i - r + 1)q^{-r-1}a\omega_{r+1, i+1}d \otimes c + q^{-1}\omega_{r+1, i}ab \otimes d \\
&\quad - q^{r+1}(2i - r - 1)d\omega_{r+1, i}a \otimes b + q^{r+2}db \otimes \omega_{r+1, i}a \\
&= (2i - r - 1)q^{r+2}d\omega_{r+2, i+1} \otimes a + (2i - r + 1)q^{-r-2}a\omega_{r+2, i+1} \otimes d \\
&\quad - q^{-r-2}ac \otimes \omega_{r+1, i+1}d - (2i - r + 1)ad\omega_{r+1, i+1} \otimes c \\
&\quad - (2i - r - 1)da\omega_{r+1, i} \otimes b + q^{r+2}db \otimes \omega_{r+1, i}a.
\end{aligned}$$

Lemma 3.5 gives $[\partial] \smile \frac{1}{2}[\partial_H^+ - \partial_H^-] = \frac{1}{2}[\partial_H^-] \smile [\partial_H^+]$, so by subtracting

$$b((2i - r - 1)\omega_{r+2, i+1} \otimes d \otimes a + b \otimes \omega_{r+1, i}a \otimes d + q^{-r-2}ac \otimes \omega_{r+1, i+1} \otimes d)$$

we get in homology

$$\begin{aligned}
& \frac{1}{2}[B(\omega_2(r, i))] \smile ([\partial_H^-] \smile [\partial_H^+]) \\
&= [(2i - r - 1)\omega_{r+2, i+1} \otimes bc + b \otimes \omega_{r+1, i} - c \otimes \omega_{r+1, i+1} \\
&\quad - q^{-1}bc^2 \otimes \omega_{r+1, i+1} - (2i - r + 1)\omega_{r+1, i+1} \otimes c \\
&\quad - (2i - r + 1)q\omega_{r+3, i+2} \otimes c - (2i - r - 1)\omega_{r+1, i} \otimes b \\
&\quad - (2i - r - 1)q^{-1}\omega_{r+3, i+1} \otimes b].
\end{aligned}$$

Using the calculus of differential forms over $\mathbb{C}[b, c]$, that is, using the fact that $[f \otimes b^j c^k] = [jfb^{j-1}c^k \otimes b] + [kfb^j c^{k-1} \otimes c]$ for $f \in \mathbb{C}[b, c]$, we obtain

$$\begin{aligned}
& \frac{1}{2}[B(\omega_2(r, i))] \smile ([\partial_H^-] \smile [\partial_H^+]) \\
&= -(2i - r)[\omega_{r+1, i} \otimes b + \omega_{r+1, i+1} \otimes c] \\
&\quad + [((2i - r - 1) - q^{-1}(r - i) - (2i - r + 1)q)\omega_{r+3, i+2} \otimes c \\
&\quad + ((2i - r - 1) - (3i - r)q^{-1})\omega_{r+3, i+1} \otimes b].
\end{aligned}$$

Applying

$$\begin{aligned}
b(\omega_{r+1, i}a \otimes b \wedge d) &= (1 - q^2)\omega_{r+3, i+1} \otimes b, \\
b(\omega_{r+1, i+1}a \otimes d \wedge c) &= (q - q^{-1})\omega_{r+3, i+2} \otimes c
\end{aligned}$$

the above finally simplifies to

$$\frac{1}{2}[B(\omega_2(r, i))] \smile ([\partial_H^-] \smile [\partial_H^+]) = -(2i - r)[\omega_{r+1, i} \otimes b + \omega_{r+1, i+1} \otimes c].$$

The claim now follows from Lemma 3.6. \square

4.2.4. Stabilisation of the spectral sequence. There is no further page of the spectral sequence to be computed - the differential on E^2 maps E_{pq}^2 to $E_{p-2, q+1}^2$, and for all p, q either one space or the other is zero. Hence:

LEMMA 4.6. For $\sigma = \sigma_{q^{-N}, 1}$, $N \geq 2$, we have $HC_n^\sigma(\mathcal{A}) \simeq \bigoplus_{p+q=n} E_{pq}^2$.

Proof. In the case $0 \in S(\lambda)$ (i.e. N odd), the E^2 page is as follows (the lines are for orientation and depict the $p = 0$ and $q = 0$ axes):

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & | & & & & \\
 0 & & 0 & & 0 & \mathbb{C}[b \otimes c] & \cdots \\
 & & | & & & & \\
 0 & \oplus_{i=0}^{N-2} \mathbb{C}[\omega_2(N-2, i)] & & \mathbb{C}[b \otimes c] & & \mathbb{C}[1] & \cdots \\
 & & | & & & & \\
 0 & \oplus_{i=0}^{N-2} \mathbb{C}[\omega_{N-1, i} \otimes b] & & \mathbb{C}[1] & & 0 & \\
 & & \oplus \mathbb{C}[b \otimes c] & & & & \\
 & & | & & & & \\
 0 & \text{-----} & HH_0^\sigma(\mathcal{A}) & \text{-----} & 0 & \text{-----} & \\
 & & | & & & & \\
 & & 0 & & & &
 \end{array}$$

Otherwise, for $0 \notin S(\lambda)$ (i.e. N even), the E^2 page is:

$$\begin{array}{ccccccc}
 0 & & 0 & & \mathbb{C}[\omega_3(N-2, \frac{1}{2}N-1)] & \mathbb{C}[\omega_2(N-2, \frac{1}{2}N-1)] & \cdots \\
 & & | & & & & \\
 0 & \mathbb{C}[\omega_3(N-2, \frac{1}{2}N-1)] & & \mathbb{C}[\omega_2(N-2, \frac{1}{2}N-1)] & & 0 & \cdots \\
 & & | & & & & \\
 0 & \oplus_{i=0}^{N-2} \mathbb{C}[\omega_2(N-2, i)] & & 0 & & 0 & \cdots \\
 & & | & & & & \\
 0 & \oplus_{i=0}^{N-2} \mathbb{C}[\omega_{N-1, i} \otimes b] & & 0 & & 0 & \\
 & & | & & & & \\
 0 & \text{-----} & HH_0^\sigma(\mathcal{A}) & \text{-----} & 0 & \text{-----} & \\
 & & | & & & & \\
 & & 0 & & & &
 \end{array}$$

□

Theorem 1.1 now follows immediately.

5. CYCLIC COHOMOLOGY

5.1. Background. In this final section we finish the proof of Theorem 1.2 which we began in Section 3 : we construct a twisted cyclic 3-cocycle ξ that pairs nontrivially with $d\mathcal{A}$, and hence represents a generator in Connes' λ -complex of $HC_{\sigma_{q^{-2}, 1}}^3(\mathcal{A}) \simeq \mathbb{C}$.

5.1.1. *Cyclic cocycles.* In view of (24), $\text{im}(\text{id} - \mathbf{t}) \subset C$ is for any cyclic \mathbb{C} -vector space a subcomplex with respect to \mathbf{b} . As Connes showed, cyclic homology can be realised as the homology of the quotient:

$$HC_\bullet(C) \simeq H_\bullet(C/\text{im}(\text{id} - \mathbf{t}), \mathbf{b}).$$

For $C = C^\sigma(\mathcal{A})$ we can dually consider Hochschild cochains $\varphi \in C^n(\mathcal{A}, (\sigma\mathcal{A})^*)$ which are twisted cyclic, that is, which satisfy

$$(27) \quad \varphi(a_0, \dots, a_n) = (-1)^n \varphi(\sigma(a_n), a_0, \dots, a_{n-1})$$

for all $a_0, \dots, a_n \in \mathcal{A}$. These form a subcomplex of $(C^\bullet(\mathcal{A}, (\sigma\mathcal{A})^*), \mathbf{b})$ whose cohomology is twisted cyclic cohomology $HC_\sigma^\bullet(\mathcal{A}) \simeq (HC_\bullet^\sigma(\mathcal{A}))^*$.

If we work with the normalised complex, then $\varphi(a_0, \dots, a_n) = 0$ whenever $a_i \in \mathbb{C}$ for some $i > 0$. For cyclic φ this property obviously extends to $i = 0$. Conversely, a Hochschild cocycle that vanishes on $1 \otimes a_1 \otimes \dots \otimes a_n$ is twisted cyclic as follows by applying it to $\mathbf{b}(1 \otimes a_0 \otimes \dots \otimes a_n)$.

5.2. **Results.** The natural question is whether the Hochschild 3-cocycle φ defined in (23) is already cyclic. We show that it is not, then construct a coboundary η by which φ differs from a cyclic cocycle.

5.2.1. *φ is not cyclic.* As remarked at the end of Section 5.1.1, cyclicity is equivalent to the condition that

$$(28) \quad \varphi(1, a_1, a_2, a_3) = 0$$

for all $a_1, a_2, a_3 \in \mathcal{A}$. Now, for all $i, j \geq 0$ we have

$$\begin{aligned} \sigma_{q, q-1}(\partial_E^+(d^j c)) \partial_F^+(a^i b) &= q^{i-2j} d^{j+1} a^{i+1}, \\ \sigma_{q^2, q^{-2}}(\partial_H^+(e_{j-i, 0, 0})) &= (i-j) q^{2(j-i)} e_{j-i, 0, 0}, \end{aligned}$$

and therefore

$$\varphi(1, e_{j-i, 0, 0}, d^j c, a^i b) = q^{-i}(i-j) \int_{[1]} e_{j-i, 0, 0} d^{j+1} a^{i+1} = q^{-i}(i-j).$$

Hence (28) fails and so φ is not cyclic.

5.2.2. *The correction term.* We make the ansatz

$$\eta(\cdot) := \int_{\gamma} \cdot \frown (\partial_H^+ \smile (\sigma_{\lambda_1, \mu_1} - \text{id}) \smile (\sigma_{\lambda_2, \mu_2} - \text{id})),$$

where \int_{γ} is a suitable twisted trace and $\lambda_1, \mu_1, \lambda_2, \mu_2$ are complex parameters.

LEMMA 5.1. *If $\lambda_1 = \lambda_2 = 1, \mu_2 = \mu_1^{-1} \neq 1$ and*

$$\int_{\gamma} = -\frac{\mu_2}{(\mu_2 - 1)^2} \int_{[bc]} \in H^0(\mathcal{A}, (\sigma_{q^{-2}, 1} \mathcal{A})^*),$$

then $\varphi + \eta$ is cyclic and as a Hochschild cocycle cohomologous to φ .

Proof. For any automorphism σ of an algebra, $\sigma - \text{id}$ is a σ -twisted derivation which is inner (it is simply the twisted commutator with $1 \in \mathcal{A}$).

Therefore its cohomology class vanishes, so η is automatically a coboundary. Furthermore, for all $i, j \geq 0$ we have

$$\begin{aligned} & \sigma_{\lambda_2, \mu_2}((\sigma_{\lambda_1, \mu_1} - \text{id})(d^j c))(\sigma_{\lambda_2, \mu_2} - \text{id})(a^i b) \\ &= q^{-i} \lambda_2^{-j} \mu_2^{-1} (\lambda_1^{-j} \mu_1^{-1} - 1) (\lambda_2^i \mu_2 - 1) d^j a^i b c, \\ & \sigma_{\lambda_1 \lambda_2, \mu_1 \mu_2}(\partial_H^+(e_{j-i, 0, 0})) \\ &= (i - j) (\lambda_1 \lambda_2)^{j-i} e_{j-i, 0, 0}. \end{aligned}$$

Hence

$$\begin{aligned} & \eta(1, e_{j-i, 0, 0}, d^j c, a^i b) \\ &= q^{-i} \lambda_2^{-j} \mu_2^{-1} (\lambda_1^{-j} \mu_1^{-1} - 1) (\lambda_2^i \mu_2 - 1) (i - j) (\lambda_1 \lambda_2)^{j-i} \int_{\gamma} e_{j-i, 0, 0} d^j a^i b c. \end{aligned}$$

Therefore, we need \int_{γ} to be a $\sigma_{q^{-2} \lambda_1 \lambda_2, \mu_1 \mu_2}$ -twisted trace for which

$$\int_{\gamma} e_{j-i, 0, 0} d^j a^i b c = - \frac{\lambda_1^{i-j} \lambda_2^i \mu_2}{(\lambda_1^{-j} \mu_1^{-1} - 1) (\lambda_2^i \mu_2 - 1)}.$$

It is easily checked that $\varphi(1, e_{i,j,k}, e_{l,m,n}, e_{r,s,t}) = \eta(1, e_{i,j,k}, e_{l,m,n}, e_{r,s,t}) = 0$ for all other $i, j, k, l, m, n, r, s, t$, since b and c are twisted central. \square

This completes the proof of Theorem 1.2.

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